Back Paper Examination : Differential Topology. BMath III

Max. Marks : 100 Time : 3 hours

Answer all questions. You may use theorems/propositions proved in the class after correctly stating them. Any other claim must be accompanied by a proof.

(1) State the preimage theorem. Let I_n denote the $n \times n$ identity matrix and X denote the set
$$X = \{A \in M_n(\mathbb{R}) \,:\, A^t A = I_n\}$$
of orthogonal matrices. Show that X is a manifold. Find $\dim(X)$ and describe the tangent space $T_A(X)$ at $A \in X$. [3+6+3+3]

(2) State Sard's theorem. Let $f, g : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ be two smooth maps. Given $\varepsilon > 0$, show that there exists $v \in \mathbb{R}^3$ such that $||v|| < \varepsilon$ and [4+10]
$$f(S^1) \cap (g(S^1) + v) = \emptyset.$$

(3) Define the notion of a Morse function on a manifold. Find the critical points of the function $f : S^1 \longrightarrow \mathbb{R}$ defined by $f(x, y) = x$. Show that f is a Morse function. [4+6+4]

(4) Let X be a manifold without boundary. Show that the tangent bundle $T(X)$ is manifold with $\dim(T(X)) = 2 \cdot \dim(X)$. [12]

(5) For what values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ (in $\mathbb{R}^3$) intersect the sphere $x^2 + y^2 + z^2 = a$ (in $\mathbb{R}^3$) transversally. Give complete justifications. [15]

(6) Let $f : X \longrightarrow Y$ be a smooth map that is one-one on a compact submanifold Z of X. Suppose that for all $x \in Z$, the derivative $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$ is an isomorphism. Show that $f : Z \longrightarrow f(Z)$ is a diffeomorphism. Further show that there exists an open set U in X with $Z \subseteq U$ such that $f(U)$ is open in Y and
$$f : U \longrightarrow f(U)$$
a diffeomorphism. The manifolds X, Y, Z are boundaryless. [5+10]

(7) Let X, Y be manifolds, $Z \subseteq Y$ a submanifold and $f : X \longrightarrow Y$ a smooth map. Under what conditions is the intersection number $I_2(f, Z)$ defined? Give the complete definition. Let $f : S^1 \longrightarrow S^2$ be the map $f(x, y) = (x, y, 0)$. Compute $I_2(f, Z)$ where
$$Z = \{(x, y, z) \in S^2 \,:\, z = 0\}$$
is the equator of S^2. [5 + 10]